Featureless Mott Insulators

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A family of the pair hopping models exhibiting the incompressible quantum liquid at fractional filling $1/m^D$ is constructed in D dimensional lattice. Except in one dimension, the lattice is the generalized edge-shared triangular lattice, for example the triangular lattice in two dimensions and tetrahedral lattice in three dimensions. They obey the new symmetry, conservation of the center-of-mass position proposed by Seidel et al..[1] The uniqueness of the ground state is proved rigorously in the open boundary condition. The finiteness of the excitation energy is calculated by the single mode approximation.

I. INTRODUCTION

The featureless Mott insulator blocks the charge transportation due to the strong electron-electron interaction and the ground state exhibits no symmetry breaking. Their existence is very rare both in the experimental systems and in the theoretical models. There are two examples in the two dimensional systems. One is the famous fractional quantum Hall effect, where the ground state is an incompressible liquid when the filling factor is $1/q$ (q = odd integer). The other example is the quantum dimer model in the two dimensional triangular lattice, where the ground state is a disordered dimer liquid that separates the excitations by a finite energy gap.

Very interestingly, Seidel et al. pointed out that these two systems actually belong to the same type of Hamiltonian.[1,2] Namely, their Hamiltonians preserve both the center-of-mass momentum and the center-of-mass position. They showed that the fractional quantum Hall system in the lowest Landau level on the torus described by the following pseudo-potential Hamiltonian[3]

$$H = \int d^2r d^2r' \nabla^2 \delta(\mathbf{r}-\mathbf{r}')\psi^+(\mathbf{r})\psi^+(\mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r}) \quad (1)$$

can map to a pair hopping model in one dimensional lattice

$$H = \sum_{R,x,y} f^*(x)f(y)C^\dagger_{R+x}C^\dagger_{R-x}\, C_{R-y}C_{R+y}$$
$$f(x) = \kappa^{3/2}\sum_n (x-nL)e^{-\kappa^2(x-nL)^2}, \quad (2)$$

where

$$L = L_xL_y/2\pi l_B^2, \quad \kappa = 2\pi l_B/L_y, \quad (3)$$

L_x and L_y are the linear dimensions of the torus, and $l_B = \sqrt{\hbar c/eB}$ is the magnetic length. It can be obviously seen that Eq.(2) describes the hopping that preserves the center-of-mass position. Two electrons annihilated at $R \pm y$ hop to $R \pm x$ with the effective hopping range $1/\kappa$ and center-of-mass position preserves at R. Due to this beautiful property, the ground states of the $1/q$ fractional quantum Hall liquid with the q-fold degeneracy can be labeled by the q different center-of-mass positions. Furthermore, Seidel et al. also showed that the ground state of Eq.(2) is a charge density wave with the amplitude $\sim e^{-c/\kappa^2}$ where c is a constant of O(1), and the energy gap is finite for any finite κ. For very small κ, where Eq.(2) becomes the long range hopping model, the charge density wave amplitude is exponentially small. One can safely say that the ground state describes a *featureless Mott insulator* without local order parameter.

The new symmetry of the center-of-mass position conservation paves a new way to search the featureless Mott insulator. Recently, inspired by the higher dimensional generalization of the quantum Hall effect[4] and the Haldane's pseudopotential Hamiltonian,[5] Chern et al. constructed a model for the incompressible liquid in the two-dimensional triangular lattice.[6] They showed that the ground state is unique without local order parameter. They also computed the excitation gap using the single mode approximation. This paper will serve as the extended version of that Letter. For this purpose, we organize the paper in the following: In the section II, we review the Haldane construction of the pseudopotential method on the two sphere. We will show explicitly that the Haldane pseudopotential Hamiltonian can map to a long-range pair hopping model with the conservation of the center of mass position. In section III, we provide the detail calculation of our previous Letter and generalize it to the SU(N) model. In section IV, we provide the detail calculation of the single mode approximation in the SU(3) case. Finally, we conclude and summarize in the section V. We also include several appendixes for the readers to follow the group-theoretical method easily.

II. ONE-DIMENSIONAL LATTICE MODEL FOR THE INCOMPRESSIBLE QUANTUM LIQUID

The quantum Hall effect can be considered on the two sphere subject to the uniform magnetic field by the U(1) magnetic monopole at the center.[5] In this case, the Laughlin wave function becomes the exact ground state of the Eq.(1). We will show that it can map to a pair hopping model with the conservation of the center-of-mass position and long range hopping integral.

In the presence of the U(1) magnetic monopole flux, the single-particle wavefunction is described by the monopole vector spherical harmonics[7,8] which can be denoted by the SU(2) $|l, m>$ state, where l can be integers or half-integers and m is the magnetic quantum number. The Landau level spectrum is given by[5]

$$E_k = \frac{\hbar^2}{2MR^2} l(l+1) \tag{4}$$

where M is the mass of the electrons and R is the radius of the sphere. If the total magnetic flux is $2S$, $l = S + k$, where k is the Landau level index. Because k is only an integer, S is either integer or half-integer. Each Landau level has $2(S + k) + 1$ degeneracy. In the lowest Landau level, $k = 0$, the single-particle wavefunction can be written as

$$\psi_i = \sqrt{\frac{(2S)!}{(S+i)!(S-i)!}} u^{S+i} v^{S-i} \tag{5}$$

where $i = -S, -S+1, .., S-1, S$ and the (u, v) is the two-dimensional complex spinor given by

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\phi/2} \\ \sin\frac{\theta}{2} e^{i\phi/2} \end{pmatrix} \tag{6}$$

where θ and ϕ parameterizing the sphere are known as the polar and azimuthal coordinates. This is the special property of the lll that the single-particle wavefunction can be completely described only by one quantum number i. Therefore, one can treat the configuration space as the 1D chain with the number of site $2S + 1$ and the lattice site is labelled by i. If the filling factor $\nu = 1$, the number of the particles $N = 2S + 1$. Then, the many-body wavefunction is the Slater determinant given by

$$\Psi = \prod_{k<l}^{N} (u_k v_l - u_l v_k) \tag{7}$$

where k and l are the particle indices. It is easy to see that Eq.(7) is the unique many-body fermionic wavefunction for $\nu = 1$.

The quantum Hall state with filling factor $\nu = 1/m$ celebrated as the Laughlin wavefunction[9] can be written as[5]

$$\Psi^m = \prod_{k<l}^{N} (u_k v_l - u_l v_k)^m \tag{8}$$

where m is an odd integer. In Eq.(8), the maximum power of u_k becomes $2mS$, indicating that the single-particle wavefunction given by the Eq.(5) is in the spin-mS state and therefore the dimension of the single- particle states (or say the lattice) is $2mS+1$. While keeping the number of particles $N = 2S + 1$ the same, Eq.(8) describes a state with filling factor

$$\nu = \frac{2S+1}{2mS+1}, \tag{9}$$

which goes to $1/m$ as S goes to infinity. Because S scales as R^2 shown in the Eq.(4), infinite S indicates to take the thermodynamic limit.

Eq.(8) is the unique ground state of the following many-body Hamiltonian

$$H_1 = \frac{1}{2} \sum_{(ij)} \sum_{k=1,\ \text{odd}}^{k \le m-2} \alpha_k P_{ij}^{2mS-k} \tag{10}$$

where α_k are positive-definite and P_{ij}^{2mS-k} are the projection operators that project two spin-mS states to the two-body states of total spin $2mS - k$ for the pair (ij). We note that k can be only the odd integers because the two-body $2mS - k$ states with odd k are antisymmetric upon particle exchanges. In Eq.(8), the term with the maximum power of $u_i u_j$ for any pair (ij) is

$$(u_i u_j)^{m(2S-1)} (u_i v_j - u_j v_i)^m, \tag{11}$$

which indicates that no two-body $2mS - k$ states for $k \le m - 2$ for any pair (ij). Therefore, Eq.(8) is the zero energy state of Eq.(10). On the other hand, because there is no two-body $2mS - k$ state for $k \le m - 2$ for any pair (ij), any ground state wavefunction χ must have the following form

$$\chi \sim f(u_1, v_1; ...; u_N, v_N) \prod_{k<l}^{N} (u_k v_l - u_l v_k)^m, \tag{12}$$

where f is the symmetric function for any pair exchange. For the single-particle wavefunction to be described by $l = mS$, the power of u_k for each particle in Eq.(12) has to be $2mS$. However, the factor

$$\prod_{k<l}^{N} (u_k v_l - u_l v_k)^m \tag{13}$$

in Eq.(12) already exhausts the quota of the power of u_k. Then, f can only be a constant. $\chi \sim \Psi_m$ and Eq.(8) is indeed the unique ground state of Eq.(10).

The SU(2) spin model of Eq.(10) can be formulated as the lattice hopping model. As we mentioned, the single-particle state can be labelled by one quantum number i. Let us denote

$$|mS, i\rangle \to c_i^\dagger |0\rangle \tag{14}$$

Then, Eq.(10) can be written as

$$\mathcal{H}_1 = \alpha \sum_n \sum_{k,q} F(n, k, q) c_k^\dagger c_{n-k}^\dagger c_{n-q} c_q. \tag{15}$$

for $m = 3$, where $F(n, k, q)$ can be obtained by the SU(2) Clebsh-Gordan coefficient

$$\begin{aligned} F(n,k,q) &= f^*(n,q) f(n,k), \\ f(n,k) &= \frac{(-1)^{2n}\sqrt{3S}(2k-n)\sqrt{12S-1}}{\sqrt{(3S-k)(3S+k)(3S+k-n)}} \\ &\times \frac{\sqrt{(6S-n-1)!(6S+n-1)!}}{\sqrt{(3S-k+n)(12S-1)!(3S-k-1)!}} \\ &\times \frac{\sqrt{(6S-1)!}}{\sqrt{(3S+k-1)!(3S+k-n-1)!(3S-k+n-1)!}} \end{aligned} \tag{16}$$

It can be easily seen that Eq.(15) is a center-of-mass conserving hopping. Two electrons annihilated with the center-of-mass position $\frac{1}{2}(k+n-k)=\frac{n}{2}$ will be created in pair with the same center-of mass position. The reason that the hopping matrix element F is n-dependent is due to the presence of the open boundary. Although F has long range hopping, the hopping matrix elements is exponentially small as the relative distance is comparable to the system size.

To analysis hopping range, let us consider the hopping with the center-of-mass to be at the origin, namely $n=0$. Then $f(n,k)=g(k)$, where

$$g(k)=\frac{2k\sqrt{3S}((6S-1)!)^2}{(k^2-9S^2)\sqrt{(12S-2)!(3S-k-1)!(3S+k-1)!}} \quad (17)$$

$g(k=0)=0$ indicates that no two electrons can be created (annihilated) at the same sites. For $S=30$, $g(k)$ is calculated in the Fig.(1). Taking $k=3S$,

$$g(3S)\sim\sqrt{S}e^{-6S\log 2}, \quad (18)$$

which has an exponentially small tail. A proper definition

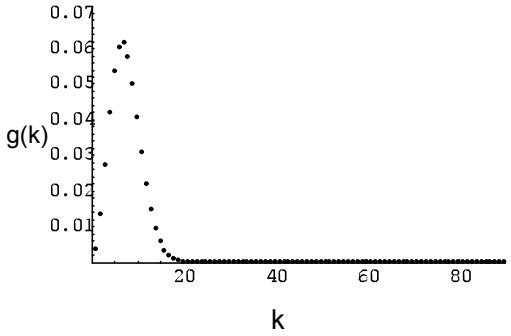

FIG. 1. The $g(k)$ versus k for $S=30$.

of the hopping range can be defined as the half width of the hump of the $|g(k)|^2$. The result that the hopping range scales with S is given in the Fig.(2), where we show the log-log relation between the hopping range and the size of the system $L=6S+1$. The trend line in the Fig.(2) is the best fit given by $\frac{1}{2}\log L-0.9$, which indicates that the hopping range scales as $L^{1/2}$.

In summary, using the SU(2) weight space, we have considered the 2D fractional quantum Hall effect as a 1D lattice model with the open boundary condition. The weight number, labeling the lattice site, is the z-component coordinate of the coherent state on the sphere. That the fractional quantum Hall state does not exhibit any density-wave order on the sphere implies that the ground state has no long range order in the 1D lattice model. On the other hand, the sphere has no edge. The

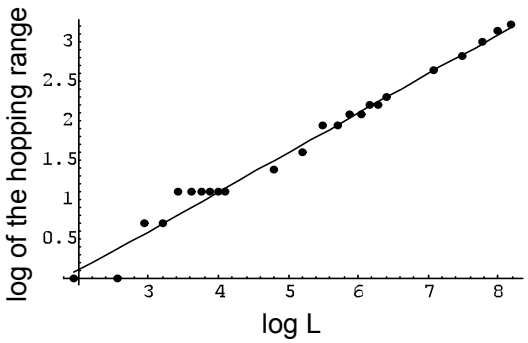

FIG. 2. A Log-Log plot of the width in $|g(k)|^2$ as a function of $L=6S+1$. We have calculated for S=0,1,2,..,10,20,30,..,100,200,300,..,600.

energy spectrum of the Eq.(15) does not include the edge modes that occur in the open 2D plane system. Thus, there is no gapless excitation corresponding in this model. Note that although the fractional quantum Hall state is the incompressible liquid shown in the experiments, theoretically the finiteness of the excitation gap is only proved by the single mode approximation[10,11].

III. HIGHER DIMENSIONAL LATTICE MODEL FOR THE INCOMPRESSIBLE QUANTUM LIQUID

There is a natural generalization of Eq.(5) to the SU(N) coherent state, which is labelled by $(N-1)$ quantum numbers *without degeneracy*. The SU(N) coherent state is known as the SU(N) $(p,0,..,0)$ multiplet. Similar to the SU(2) case, we will construct a family of the faithful lattice models for the incompressible quantum liquid using the SU(N) coherent states. We shall start with the SU(3) for the pedagogical purpose.

The SU(3) coherent states are given by

$$\sqrt{\frac{p!}{m_1!m_2!m_3!}}u^{m_1}v^{m_2}w^{m_3},\ m_1+m_2+m_3=p \quad (19)$$

which forms the multiplet described the SU(3) $(p,0)$ representation and m_i are integers. The (u,v,w) in Eq.(19) is the complex spinor which can be represented by

$$\psi_\alpha(z_1,z_2)=\begin{pmatrix}u\\v\\w\end{pmatrix}=\frac{1}{\sqrt{1+\bar{z}_1z_1+\bar{z}_2z_2}}\begin{pmatrix}1\\z_1\\z_2\end{pmatrix} \quad (20)$$

where z_i are the complex numbers to parameterize CP^2 and α is the spinor index. As shown in the Appendix, the SU(3) $(p,0)$ states is the lowest Landau level in the quantum Hall problem in CP^2. The SU(3) Cartan subalgebra contains two generators T_3 and T_8. The states of

Eq.(19) acquire the coordinates in the T_3 and T_8 space, which form the two-dimensional triangular lattice shown in the Fig.3. The number of sites of the two-dimensional

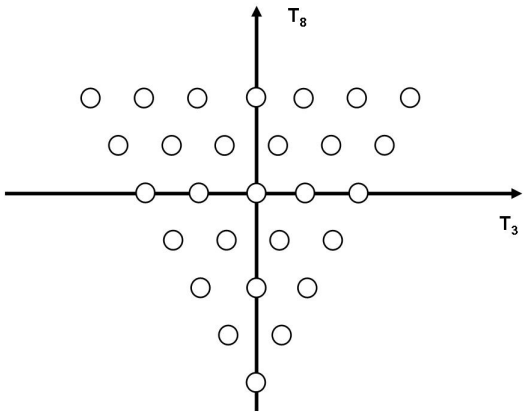

FIG. 3. The weight space of $(6,0)$. T_3 and T_8 are the Gel-Mann matrices forming the Cartan subalgebra.

lattice representing the SU(3) $(p,0)$ multiplet is given by

$$d(p) = \frac{1}{2}(p+1)(p+2) \tag{21}$$

If the number of electrons $N = d(p)$, the many-body fermionic wavefunction is the Slater determinant given by

$$\Psi = \begin{vmatrix} u_1^p & u_1^{p-1}v_1 & . & . & w_1^p \\ u_2^p & u_2^{p-1}v_2 & . & . & w_2^p \\ . & . & . & . & . \\ . & . & . & . & . \\ u_N^p & u_N^{p-1}v_N & . & . & w_N^p \end{vmatrix} \tag{22}$$

up to the normalization constant. It is also not hard to see that Eq.(22) is the unique fermionic many-body wavefunction when $N = d(p)$. Next, let us consider the natural generalization of Eq.(8) written by

$$\Psi^m = \begin{vmatrix} u_1^p & u_1^{p-1}v_1 & . & . & w_1^p \\ u_2^p & u_2^{p-1}v_2 & . & . & w_2^p \\ . & . & . & . & . \\ . & . & . & . & . \\ u_N^p & u_N^{p-1}v_N & . & . & w_N^p \end{vmatrix}^m \tag{23}$$

Because the highest power of u for each particle in Eq.(23) is mp, the new coherent state is described by the SU(3) $(mp,0)$ multiplet. Therefore, Eq.(23) is the many-body state with the filling factor

$$\nu = \frac{(p+1)(p+2)}{(mp+1)(mp+2)} \tag{24}$$

which becomes $1/m^2$ in the thermodynamic limit. We propose the following SU(3) spin Hamiltonian

$$H = \frac{1}{2}\sum_{i\neq j}\sum_{q=1,\ \text{odd}}^{q\leq m-2} \kappa_q\ P_{ij}^{(2mp-2q,q)}. \tag{25}$$

where the operator $P_{ij}^{(2mp-2q,q)}$ operates on the direct product states of two spins i and j and projects them onto the $(2mp-2q,q)$ states, and κ_q are positive-definite. The SU(3) two-spin states are the generalization of the SU(2) angular momentum addition. The direct product of two SU(3) multiplets can be also block-diagonalized such that each block is described a SU(3) multiplet denoted by two integers (a,b). In this case, the direct product of two SU(3) $(mp,0)$ multiplets can computed as

$$(mp,0)\times(mp,0)|_{\rm a} = \bigoplus_{q=1,\text{odd}}^{mp}(2mp-2q,q) \tag{26}$$

where the subscript "a" denotes the antisymmetric combination and k are the odd integers. Furthermore, every SU(3) multiplet can be also block-diagonalized by its SU(2) subgroup. In our case, the SU(3) $(p,0)$ multiplet can be decomposed as

$$(p,0) = \bigoplus_{k=0}^{p}\frac{k}{2} \tag{27}$$

which can be easily checked by the counting the dimensionality.

The Eq.(23) is the zero-energy state of Eq.(25). We first look at the two-spin state of Eq.(22). Because all sites are occupied, the SU(3) two-spin state with the maximum SU(2) quantum number is the $(2p-2,1)$ multiplet. Because Eq.(23) is the (m-fold) product of the Eq.(22), the SU(3) multiplet with the maximum SU(2) quantum number that the direct product of m $(2p-2,1)$ multiplets can yield is $(2mp-2m,m)$. Therefore, due to the SU(3) symmetry, the two-spin states in Eq.(23) does not contain the multiplets $(2mp-2k,k)$ for $k \leq m-2$. The Hamiltonian is the SU(3) generalization of the Eq.(10). Our generalized Laughlin wavefunction of Eq.(23) is the zero-energy state of the generalized Hamiltonian of Eq.(25). Moreover, it is also the *unique* ground state. In the Letter[6], we have demonstrated rigorously that Eq.(23) is the *unique* ground state of the Hamiltonian of Eq.(25). Some supplemental details will be discussed in the Appendix.

Eq.(25) can be written as a lattice hopping model. Similar to Eq.(14), the electron creation operator can be defined as

$$|(mp,0);j,j_3> \rightarrow c^\dagger_{(j,j_3)}|0>, \tag{28}$$

where the states are denoted by the quantum numbers of the SU(2) subgroup. This coordinate system is equivalent to the quantum numbers of T_3 and T_8, since there is no degeneracy on the lattice. Using Eq.(28), the Hamiltonian for $m=3$ can be expressed by

$$H = \kappa \sum_{j,L,L_3}\sum_{l,l_3}\sum_{k,k_3} F^{j,L,L_3}_{l,l_3}F^{j,L,L_3}_{k,k_3} \\ c^\dagger_{(l,l_3)}c^\dagger_{(j-l+\frac{1}{2},L_3-l_3)}c_{(j-k+\frac{1}{2},L_3-k_3)}c_{(k,k_3)}, \tag{29}$$

where κ is a positive number and F^{j,L,L_3}_{l,l_3} is the SU(3) Clebsh-Gordan coefficient from two $(3p,0)$ multiplets to

the $(6p-2,1)$ subspace shown in Eq.(26). In Eq.(29) the center-of-mass position is conserved at $(j+1/2, L_3)$ in the pair-annihilation and pair-creation process.

The Hamiltonian of Eq.(29) also describes a long-ranged hopping process. The F^{j,L,L_3}_{l,l_3} can be computed exactly as[12,13]

$$F^{j,L,L_3}_{l,l_3} = -\frac{(3p)!}{(6p)!}\sqrt{\frac{(6p-1)2!(6p+L-j-2l-\frac{1}{2})!}{(3p-2j+2l-1)!}}$$
$$\times \sqrt{\frac{(6p-L-j-\frac{3}{2})!(2j+1)!}{(3p-2l)!}}\sqrt{\frac{(2j-2l+2)!(2l+1)!}{(2j-2l)!(2l-1)!}}$$
$$\times (-1)^{\frac{1}{2}+j+L_3-2l}\sqrt{2L+1}\begin{pmatrix} j+\frac{1}{2}-l & l & L \\ L_3-l_3 & l_3 & L_3\end{pmatrix}$$
$$\times\Big(\frac{1}{2l}\begin{Bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ t & l & j \\ t+\frac{1}{2} & l & L\end{Bmatrix} - \frac{1}{2t+1}\begin{Bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ t+\frac{1}{2} & l-\frac{1}{2} & j \\ t+\frac{1}{2} & l & L\end{Bmatrix}\Big), \quad (30)$$

where $t=j-l$ and the Wigner 6-j and 9-j symbols are explicitly used. In Fig.4, we plot the F^{j,L,L_3}_{l,l_3} for $p=300$, $j=3p-\frac{3}{2}$, $L=3p-1$, and $l=\frac{3p}{2}$ as the function of l_3. It is the pair-hopping integral with the relative

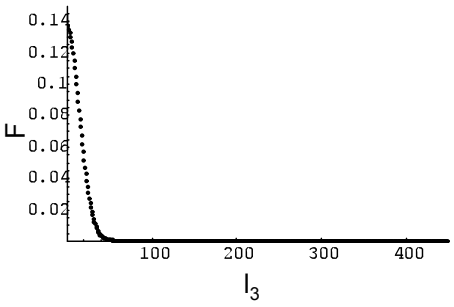

FIG. 4. $F^{p,3p-\frac{3}{2},3p-1,0}_{\frac{3p}{2},l_3}$, $p=300$, $l_3=1..\frac{3p}{2}-1$

distance $\sqrt{3+4l_3^2}$. Because two electrons are not on the same rows, F^{j,L,L_3}_{l,l_3} is not zero at $l_3=0$. The hopping range can be also defined as the half-width of $|F|^2$. In Fig.5, we show the log-log relation between the hopping range and p. The result suggests that the hopping range scales as $\sqrt{p}$. Similar to the SU(2) case, when the relative distance is the same order of the system size, the pair-hopping integral is exponentially decayed as

$$F^{p,3p-\frac{3}{2},3p-1,0}_{\frac{3p}{2},\frac{3p}{2}} \sim \sqrt{6p(6p-1)}\; e^{-3p\log 2} \quad (31)$$

In the later section, we will show the existence of the finite excitation gap within the single-mode approximation. In the presence of the energy gap, the uniqueness of the ground state implies that it is an incompressible quantum liquid. Here we shall consider the more general case.

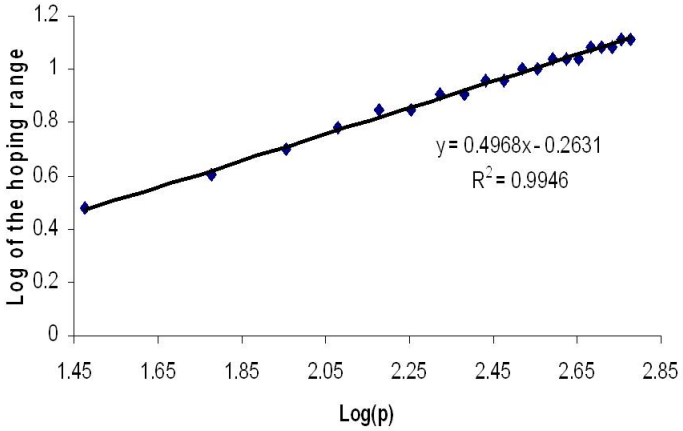

FIG. 5. A log-log plot of the hopping range to p from 30 to 600. The straight line is the best fit. The vertical axis is the log of the hopping range and the horizontal one is $\log p$. The hopping range scales as $p^{\frac{1}{2}}$

Our current formalism can be generalized to the SU(N) case very easily. The SU(N) fundamental spinor given by

$$\psi_\alpha(\{z_i\}) = \begin{pmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ u_N\end{pmatrix} = \frac{1}{\sqrt{1+\sum_{i=1}^N \bar{z}_i z_i}}\begin{pmatrix} 1 \\ z_1 \\ z_2 \\ \cdot \\ \cdot \\ z_N\end{pmatrix} \quad (32)$$

can be used to parameterized CP^{N-1}, where z_i are complex numbers. The SU(N) coherent states given by

$$\sqrt{\frac{p!}{\prod_{i=1}^N m_i!}}u_1^{m_1}u_2^{m_2}\cdots u_N^{m_N},\ \sum_{i=1}^N m_i = p \quad (33)$$

are described by the SU(N) $(p,0,0,..,0)$ multiplet. A general SU(N) multiplet is labelled by $N-1$ integers $(n_1,..,n_N)$. The rank of the SU(N) group is $N-1$. Therefore, Eq.(33) represents a $(N-1)$-dimensional lattice. As we have seen, it is a triangular lattice in two dimensions, and in three dimensions it is a tetrahedral lattice, and so on. One of the common features is that they are all frustrated lattices. The number of the lattice is also related to p, namely the maximum power of the u_i, given by the following formula

$$d_N(p) = \frac{1}{(N-1)!}(p+1)(p+2)\cdots(p+N-1). \quad (34)$$

When the number of the electrons is equal to the number of sites, $N=d_N(p)$, the fermionic many-body wavefunc-

tion is given by

$$\Psi_N = \begin{vmatrix} u_{11}^p & u_{11}^{p-1}u_{21} & . & . & u_{N1}^p \\ u_{12}^p & u_{12}^{p-1}u_{22} & . & . & u_{N2}^p \\ . & . & . & . & . \\ . & . & . & . & . \\ u_{1N}^p & u_{1N}^{p-1}u_{2N} & . & . & u_{NN}^p \end{vmatrix} \tag{35}$$

where u_{ij} is the i^{th} component of spinor in Eq.(32) for the j^{th} electron. Similarly, let us consider the wavefunction Ψ_N^m. Because the maximum power of u_1 for each electron becomes mp, it describes a state with fractional filling factor

$$\nu_N = \frac{(p+1)(p+2)\cdot\cdot(p+N-1)}{(mp+1)(mp+2)\cdot\cdot(mp+N-1)} \tag{36}$$

which approaches to $1/m^{N-1}$ or $1/m^D$, where D is the dimensionality, in the thermodynamic limit.

The Hamiltonian for the Ψ_N^m is the zero-energy state is given by

$$H = \frac{1}{2}\sum_{i\neq j}\sum_{q=1,\ \text{odd}}^{q\leq m-2} \kappa_q\ P_{ij}^{(2mp-2q,q,0,0,..,0)_N}. \tag{37}$$

where κ_q are positive and $P_{ij}^{(2mp-2q,q,0,0,..,0)_N}$ projects two SU(N) $(mp,0,..,0)_N$ spins to the $(2mp-2q,q,0,0,..,0)_N$ spin. The argument for Ψ_N^m to be the zero-energy state is quite similar to the SU(3) case. We also leave the proof for the non-degeneracy of the ground state in the Appendix.

Each the SU(N) coherent state is non-degenerate. States in the SU(N) $(p,0,..,0)_N$ multiplet can be labelled by its SU(N-1) subgroup following the relation

$$(p,0,0,..,0)_N = \bigoplus_{x=0}^{p}(x,0,..,0)_{N-1} \tag{38}$$

From Eq.(27) and Eq.(38), there is an inductive relation to label the states in the SU(N) $(p,0,..,0)_N$ multiplet. In other words, they can be labelled by the SU(N-1), SU(N-2), .., and SU(2) subgroups namely

$$|(mp,0,0,..,0)_N; x_1,x_2,..,x_{N-1}> \tag{39}$$

where x_i denote the SU(i) $(x_i,0,..,0)_i$ multiplets. Using this coordinate system, the electron creation operator is defined by

$$|(p,0,0,..,0)_N; x_1,x_2,..,x_{N-1}> \rightarrow c^\dagger_{(x_1,x_2,..,x_{N-1})}|0> \tag{40}$$

Let us denote $\vec{x} = (x_1,x_2,..,x_{N-1})$. Then, similar to the Eq.(25), Eq.(37) can be written as

$$H = \kappa \sum_{\vec{R}\in(2mp-2,1,0,..,0)_N}\ \sum_{\vec{y},\vec{x}} F(\vec{R},\vec{y})F(\vec{R},\vec{x})c^\dagger_{\vec{y}}c^\dagger_{\vec{R}-\vec{y}}c_{\vec{R}-\vec{x}}c_{\vec{x}} \tag{41}$$

for $m=3$, where $F(\vec{R},\vec{x})$ is the Clebsh-Gordan coefficient from two SU(N) $(3p,0,0,..,0)_N$ spins to the $(6p-2,1,0,..,0)_N$ spin. The only tricky point in Eq.(41) is the center-of-mass $\vec{R}$. Because the SU(N) $(6p-2,1,0,0,..,0)_N$ multiplet contains degeneracy in its states, the $(N-1)$-dimensional vector $\vec{R}$ is the function of the quantum numbers of the subgroups and the Cartan subalgebra. Hence, we have shown that the general Hamiltonian so that the general Laughlin states are the non-degenerate ground state. It is obvious to see that it preserves the conservation law of the center-of-mass position.

So far there is no efficient way to calculate the Clebsh-Gordan coefficients for the general SU(N)×SU(N) to SU(N) case. In the SU(3) case, only the Clebsh-Gordan coefficients for small multiplets can be calculated numerically. However, $F(\vec{R},\vec{x})$ can be obtained analytically in general because

$$\begin{aligned}&(mp,0,0,..,0)_N \times (mp,0,0,..,0)_N \\ &\quad = \bigoplus_{q=1}^{mp}(2mp-2q,q,0,..,0)_N,\end{aligned} \tag{42}$$

where the right hand side contains no repeated multiplets. We leave this important mathematical problem to mathematical physicists and conjecture that the pair-hopping range defined by the half width of the $|F(\vec{R},\vec{x})|^2$ is also long-ranged scaling as $\sqrt{p}$ with an exponential decay tail for the general case.

IV. SINGLE MODE APPROXIMATION

In the SU(2) case, Girvin et al.[10,11] showed that for any liquid ground state in the lowest Landau level, the density fluctuation costs finite energy, which implies incompressibility, within the single mode approximation. In this section, we generalize their result to the SU(N) model.

As mentioned in the earlier section, our lattice model corresponds to the fractional quantum Hall effect in CP^{N-1} which is subject to the background U(1) magnetic field with the quantization n. The single-particle Lagrangian in the lowest Landau level is given by

$$L = -in\bar{\psi}_\alpha\frac{d\psi_\alpha}{dt} \tag{43}$$

Using Eq.(32), in the flat-space limit Eq.(43) can be written as

$$L = \sum_{k=1}^{N-1} nx_k\dot{y}_k - ny_k\dot{x}_k = A_j\dot{X}_j \tag{44}$$

where we set $|z_k| \ll 1$ and $z_k = x_k + iy_k$ for $k=1$ to $N-1$ and A_j and X_j are the $2(N-1)$-dimensional vector potential and the position vector ($\{x_k,y_k\}$) respectively. From Eq.(44), the single-particle orbit in the lowest Landau level can be obtained as

$$\Phi_{\{l_k\}}(\{z_k\}) = \prod_{k=1}^{N-1}\frac{1}{\sqrt{2\pi 2^{l_k}l_k!}}z_k^{l_k}e^{\frac{-|z_k|^2}{4}} \tag{45}$$

where l_k are non-negative integers. We recognize that Eq.(45) is the product of the $(N-1)$ lowest-Landau-level wavefunctions in two space dimensions. Thus, in the flat-space limit, the lowest Landau level in CP^{N-1} becomes the direct product of $(N-1)$ quantum Hall effects in two dimensions. It can be seen by the non-commutative algebra as well. From Eq.(44)

$$[x_k, y_k] = -\frac{i}{n} \text{ for } k = 1 \text{ to } N-1 \tag{46}$$

which indicates the there are $N-1$ independent non-commutative planes. In the rest of the section, we present the results for $N = 3$. The formalism can be generalized to any N easily. We also note that in the rest of the section N is the symbol for the number of particles.

In the single-mode approximation (SMA), the variational excitation energy is given by

$$\Delta(k_1, k_2) = f(k_1, k_2)/s(k_1, k_2), \tag{47}$$

where k_i are the complex wave numbers in the i^{th} quantum Hall plane and $f(k_1, k_2)$ and $s(k_1, k_2)$ are the oscillator strength and the static autocorrelation function given by

$$f(k_1, k_2) = \frac{1}{N} < \Psi_m|[\rho^\dagger_{(k_1,k_2)}, [H, \rho_{(k_1,k_2)}]]|\Psi_m > \tag{48}$$

$$s(k_1, k_2) = \frac{1}{N} < \Psi_m|\rho^\dagger_{(k_1,k_2)}\rho_{(k_1,k_2)}|\Psi_m > \tag{49}$$

respectively, where $\rho_{(k_1,k_2)}$ is the density operator. Here we use adopt Girvin et al.'s notation[10,11] which is a little bit different from our previous Letter[6]. In the lowest Landau level, both $f(k_1, k_2)$ and $s(k_1, k_2)$ should be treated carefully because coordinates do not mutually commute. Particularly, the kinetic energy vanishes and they should be redefined by

$$\bar{f}(k_1, k_2) = \frac{1}{N} < \Psi_m|[\bar{\rho}^\dagger_{(k_1,k_2)}, [\bar{V}, \bar{\rho}_{(k_1,k_2)}]]|\Psi_m> \tag{50}$$

$$\bar{s}(k_1, k_2) = \frac{1}{N} <\Psi_m|\bar{\rho}^\dagger_{(k_1,k_2)}\bar{\rho}_{(k_1,k_2)}|\Psi_m> \tag{51}$$

where $\bar{\rho}$ is the projected density operator and $\bar{V}$ is the projected potential energy in the lowest Landau level, which are respectively given by

$$\bar{\rho}_{(k_1,k_2)} = \sum_{j=1}^{N} e^{-ik_1 \frac{\partial}{\partial z_{1j}}} e^{-ik_2 \frac{\partial}{\partial z_{2j}}} e^{-\frac{ik_2^*}{2} z_{2j}} e^{-\frac{ik_1^*}{2} z_{1j}}$$

$$\bar{V}=\frac{1}{2}\int \frac{d^2q_1 d^2q_2}{(2\pi)^4} v(q_1, q_2)(\bar{\rho}^\dagger_{(q_1,q_2)}\bar{\rho}_{(q_1,q_2)} - \rho e^{-\frac{|q_1|^2+|q_2|^2}{2}}) \tag{52}$$

where ρ is the average density. In Eq.(52), $v(q_1, q_2)$ is required to be positive indicating the repulsive interaction to ensure the excitation energy to be positive. Using the algebra for the density operator

$$[\bar{\rho}_{(k_1,k_2)}, \bar{\rho}_{(q_1,q_2)}] = (e^{\frac{k_1^* q_1 + k_2^* q_2}{2}} - e^{\frac{k_1 q_1^* + k_2 q_2^*}{2}})\bar{\rho}_{(k_1+q_1, k_2+q_2)}, \tag{53}$$

$\bar{f}(k_1, k_2)$ can be easily computed as

$$\bar{f}(k_1, k_2) = \frac{1}{2}\sum_{q_1,q_2} v(q_1, q_2)(e^{\frac{q_1^* k_1 + q_2^* k_2}{2}} - e^{\frac{q_1 k_1^* + q_2 k_2^*}{2}})[$$
$$\bar{s}(q_1, q_2)e^{-\frac{|k_1|^2+|k_2|^2}{2}}(e^{-\frac{k_1^* q_1 + k_2^* q_2}{2}} - e^{-\frac{k_1 q_1^* + k_2 q_2^*}{2}})$$
$$+\bar{s}(k_1+q_1, k_2+q_2)(e^{\frac{k_1^* q_1 + k_2^* q_2}{2}} - e^{\frac{k_1 q_1^* + k_2 q_2^*}{2}})] \tag{54}$$

A direct expansion shows that $\bar{f}(k_1, k_2)$ vanishes in the fourth order in k. To show the necessary condition for existence of the excitation gap, we have to demonstrate that $\bar{s}(k_1, k_2)$ vanishes in the same order in k. Then, because of the isotropy between k_1 and k_2

$$\Delta(k_1, k_2) = \frac{a|k_1|^4 + b|k_1|^2|k_2|^2 + a|k_2|^4}{c|k_1|^4 + d|k_1|^2|k_2|^2 + c|k_2|^4} \tag{55}$$

remains finite as k approaches to zero in any direction, where a, b, c , and d are constants.

The asymptotic behavior of $\bar{s}(k_1, k_2)$ can be analyzed by relating with the radial distribution function $g(\vec{r})$ by

$$s(\vec{k}) = 1 + \rho \int d^4 r e^{-i\vec{k}\cdot\vec{r}}[g(\vec{r}) - 1] + \rho(2\pi)^4\delta^4(\vec{k}) \tag{56}$$

where $\vec{k} = (\mathrm{Re}k_1, \mathrm{Im}k_1, \mathrm{Re}k_2, \mathrm{Im}k_2)$ and $\vec{r} = (x_1, y_1, x_2, y_2)$ are real vectors. For filling factor $\nu = 1/m^2$ we obtained

$$\rho[g(\vec{r}) - 1] = \frac{m^2}{4\pi^2}\sum_{l_1,l_2=0}^{\infty} \frac{(\frac{r_1^2}{2})^{l_1}(\frac{r_2^2}{2})^{l_2}}{l_1! l_2!} e^{-\frac{r_1^2+r_2^2}{2}}($$
$$< n_{l_1 l_2} n_{00} > - < n_{l_1 l_2} >< n_{00} > - \frac{1}{m^2}\delta_{(l_1 l_2)(00)}) \tag{57}$$

where $r_1^2 = x_1^2 + y_1^2$ and $r_2^2 = x_2^2 + y_2^2$, and $n_{l_1 l_2} = c^\dagger_{l_1 l_2} c_{l_1 l_2}$ where $c^\dagger_{l_1 l_2}$ is the electron creation operator in the orbit (l_1, l_2). Establish the relation between $\bar{s}(k_1, k_2)$ and $s(k_1, k_2)$

$$\bar{s}(k_1, k_2) = s(k_1, k_2) - (1 - e^{-\frac{k_1^2+k_2^2}{2}}), \tag{58}$$

and define

$$M_{n_1 n_2} \equiv \int d^2 r_1 d^2 r_2 (\frac{r_1^2}{2})^{n_1}(\frac{r_2^2}{2})^{n_2}\rho[g(r_1, r_2) - 1] \tag{59}$$

One can compute easily that

$$M_{00} = m^2(< N n_{00} > - < N >< n_{00} >) - 1$$
$$M_{10} = m^2(< (L_1 + N) n_{00} > - < L_1 + N >< n_{00} >) - 1$$
$$M_{01} = m^2(< (L_2 + N) n_{00} > - < L_2 + N >< n_{00} >) - 1 \tag{60}$$

where $N = \sum_{l_1 l_2} n_{l_1 l_2}$ is the total number of particles and $L_i = \sum_{l_1 l_2}(l_i) n_{l_1 l_2}$ are the total angular momentum on the i^{th} quantum Hall plane. Because of the conservation of the number of particles and the angular momentum, their fluctuation is zero, so $M_{00} = M_{10} = M_{01} = -1$. Then, the second order of k in $\bar{s}(k_1, k_2)$ vanishes. The asymptotic behavior of $\bar{s}(k_1, k_2)$ indeed scales as the fourth order in k. Eq.(55) holds true.

The current analysis relies on the transformation from CP^{N-1} to $R^{2(N-1)}$, namely the flat-space limit. The former one is a compact space whose volume is finite without boundary. The later one is a non-compact space whose volume is infinite but the exponential term in Eq.(45) sets the natural boundary. An importance question is whether the structure of the energy spectrum is preserved in the transformation. We believe that the energy spectrum is not a one-to-one mapping, because in $R^{2(N-1)}$ there are certainly gapless edge excitations. On the other hand, in CP^{N-1} there is no edge excitation due to the lack of the boundary. However, the structure of the bulk excitation is preserved because we do not introduce any flux which generates the electro-motive force to close/open a gap in the transformation. Therefore, the existence of the finite excitation gap in $R^{2(N-1)}$ implies that our lattice model also has a finite excitation gap. Besides the edge modes, whether or not there are other gapless bulk excitations can not be answered by the current approximation in $R^{2(N-1)}$. If there are gapless bulk excitations in $R^{2(N-1)}$, it should be also true in our lattice model. As far as we can say, within the single mode approximation, we conclude that our unique ground state describes an incompressible quantum liquid.

V. DISCUSSION AND CONCLUSION

Guided by the new symmetry of the center-of-mass position, we construct a family of the models to describe the fractionally-filled incompressible liquid in *any* dimension. They are long range pair hopping model in the frustrated lattice in $d \geq 2$. We prove rigorously the uniqueness of the ground state in the open boundary condition using the group-theoretical method. We also compute the energy gap using the single mode approximation, which is still the best analytical method to show the finiteness of the energy gap in the fractional quantum Hall effect. Since the model is highly related to the higher dimensional generalization of the quantum Hall effect, one can generalize our models in different topological structures, for example, the torus and discuss the possibility of the fractionally-charged excitation for the future explorations.

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Appendix A: Proof of the non-degeneracy of the zero-energy state

In this section, we present the rigorous proof of the uniqueness of the Ψ_m as the zero-energy state of the Hamiltonian in the Eq.(37). The key procedure has been outlined in our previous Letter[6].

For simplicity, we focus on $m = 3$ and set p to be an odd integer. In this case, Eq.(37) has the following form

$$H = \frac{\kappa_1}{2} \sum_{i \neq j} P_{ij}^{(6p-2,1,0,0,..,0)} \tag{A1}$$

The Hamiltonian contains the term which projects two SU(N) spins in the $(3p, 0, .., 0)_N$ to the $(6p-2, 1, 0, .., 0)_N$ subspace. The direct product of two SU(N) $(3p, 0, .., 0)_N$ spins can be decomposed by the SU(N) subspace given in the Eq.(42). Particularly, when considering the antisymmetric combination, the complete set of spaces reduces to

$$\begin{aligned}(mp, 0, 0, .., 0)_N \times (mp, 0, 0, .., 0)_N|_a \\ = \bigoplus_{q=1,\ \text{odd}}^{mp} (2mp - 2q, q, 0, .., 0)_N,\end{aligned} \tag{A2}$$

where only odd q are allowed. If χ is the zero-energy state of Eq.(A1), there is no $(6p-2, 1, 0, .., 0)$ component in its two-spin spectrum. In other words,

$$(P_{ij}^{(6p-6,3,0,..,0)_N} + P_{ij}^{(6p-10,5,0,..,0)_N} + .. + P_{ij}^{(0,3p,0,..,0)})\chi = \chi \tag{A3}$$

for any pair (ij). Eq.(A3) constraints the symmetry properties in the zero-energy state. We note that the state in the SU(N) $(3p, 0, .., 0)_N$ multiplet can be written as the symmetric product of $3p$ SU(N) fundamental spinor in the Eq.(32). For particle j, it is given by

$$\psi_j^{\alpha_{j1}} \psi_j^{\alpha_{j2}} \psi_j^{\alpha_{j3}} .. \psi_j^{\alpha_{j,3p}} \tag{A4}$$

where α_{jk} runs from 1 to N. Eq.(A4) is the alternative way of expressing Eq.(33). Using Eq.(A4) as the basis, χ in general can be written as

$$\chi = \sum_{\{\alpha_{jn}=1\}}^{N} C(\{\alpha_{jn}\}) \prod_{j=1}^{d(p)} \prod_{n=1}^{3p} \psi_j^{\alpha_{jn}}. \tag{A5}$$

where the wavefunction $C(\{\alpha_{jn}\})$ satisfies the following Schrödinger equation from the Eq.(A3)

$$\begin{aligned}&C(..,\{\alpha_i\}..,\{\alpha_j\},..) \\ &= \sum_{\{\beta_i\},\{\beta_j\}} [A_3(\{\alpha_i\},\{\alpha_j\};\{\beta_i\},\{\beta_j\}) \\ &+ A_5(\{\alpha_i\},\{\alpha_j\};\{\beta_i\},\{\beta_j\}) + ... \\ &+ A_{3p}(\{\alpha_i\},\{\alpha_j\},\{\beta_i\},\{\beta_j\})] C(..,\{\beta_i\},..,\{\beta_j\},..)\end{aligned} \tag{A6}$$

for any pair (ij), where A_q are the tensors for the projection operator $P_{ij}^{(6p-2q,q,0,..,0)_N}$. A_q does the following symmetric operations

1. q of the $3p$ indices of particle i is made antisymmetric to q indices of particle j.

2. the rest of the indices of particle i is made totally symmetric to the rest of the indices of particle j.

There is degree of freedom to choose which pair of indices is made antisymmetric in the symmetry operation given above. For example, A_3 can be written as

$$\begin{aligned}&A_3(\{\alpha_i\},\{\alpha_j\};\{\beta_i\},\{\beta_j\})\\&=\frac{1}{N_3}(\delta^{\alpha_{i1}}_{\beta_{i1}}\delta^{\alpha_{j1}}_{\beta_{j1}}-\delta^{\alpha_{i1}}_{\beta_{j1}}\delta^{\alpha_{j1}}_{\beta_{i1}})(\delta^{\alpha_{i,p+1}}_{\beta_{i,p+1}}\delta^{\alpha_{j,p+1}}_{\beta_{j,p+1}}-\delta^{\alpha_{i,p+1}}_{\beta_{j,p+1}}\delta^{\alpha_{j,p+1}}_{\beta_{i,p+1}})\\&(\delta^{\alpha_{i,2p+1}}_{\beta_{i,2p+1}}\delta^{\alpha_{j,2p+1}}_{\beta_{j,2p+1}}-\delta^{\alpha_{i,2p+1}}_{\beta_{j,2p+1}}\delta^{\alpha_{j,2p+1}}_{\beta_{i,2p+1}})(\delta^{\alpha_{i2}}_{\beta_{i2}}..\delta^{\alpha_{ip}}_{\beta_{ip}}\delta^{\alpha_{ip+2}}_{\beta_{i,p+2}}..\delta^{\alpha_{i,2p}}_{\beta_{i,2p}}\\&\delta^{\alpha_{i,2p+2}}_{\beta_{i,2p+2}}..\delta^{\alpha_{i,3p}}_{\beta_{i,3p}}\delta^{\alpha_{j2}}_{\beta_{j2}}..\delta^{\alpha_{jp}}_{\beta_{jp}}\delta^{\alpha_{jp+2}}_{\beta_{j,p+2}}..\delta^{\alpha_{j,2p}}_{\beta_{j,2p}}\delta^{\alpha_{j,2p+2}}_{\beta_{j,2p+2}}..\delta^{\alpha_{j,3p}}_{\beta_{j,3p}}\\&+\text{sym.}),\end{aligned}\tag{A7}$$

where N_3 is the normalization constant. Consequently, for a particular pair (ij), one can arrange the antisymmetric pairs so that C becomes $-C$ by the following independent exchanges

$$\begin{aligned}&(\alpha_{i1}..\alpha_{ip})\leftrightarrow(\alpha_{j1}..\alpha_{jp})\\&(\alpha_{i,p+1}..\alpha_{i,2p})\leftrightarrow(\alpha_{j,p+1}..\alpha_{j,2p})\\&(\alpha_{i,2p+1}..\alpha_{i,3p})\leftrightarrow(\alpha_{j,2p+1}..\alpha_{j,3p})\end{aligned}\tag{A8}$$

If the symmetry property of C shown above can be made true *simultaneously* for *any* pair (ij). The argument that Ψ_m is the unique zero-energy state can be given as the following. Let us consider the independent exchange of the first group of p indices while keeping others fixed. C becomes $-C$ has to be established in any pair (ij). Because the number of particle is $N=d(p)$ is exactly equal to the total number of states that p indices represents, C is proportional to

$$C\sim\epsilon\{(\alpha_{j1}..\alpha_{jp})\}\tag{A9}$$

where $\epsilon\{(\alpha_{j1}..\alpha_{jp})\}$ is the rank $d(p)$ tensor with respect to the group exchange. Similarly, the second and third properties in the Eq.(A8) leads to

$$C\sim\epsilon\{(\alpha_{j1}..\alpha_{jp})\}\epsilon\{(\alpha_{j,p+1}..\alpha_{j,2p})\}\epsilon\{(\alpha_{j,2p+1}..\alpha_{j,3p})\}\tag{A10}$$

Using Eq.(A10), any zero-energy state χ is proportional to Ψ_m.

Now, we shall prove that Eq.(A8) can indeed be made true for all pairs (ij) simultaneously. Let us assume that there exists a ground state solution whose C does not satisfy Eq.(A8) for pair (kl). It means there is at least one group exchange, say $\{\alpha_{k1}..\alpha_{kp}\}\leftrightarrow\{\alpha_{l1}..\alpha_{lp}\}$, so that C does not follow Eq.(A8). However, since the wavefunction still has to satisfy Eq.(A6) for (k,l), one can write

$$C=\sum_{q=3,\text{odd}}^{3p}C_q,\tag{A11}$$

where C_q is the component of C that is odd with respect to exchange of *exactly* q pair of indices between particle k and l and even with respect to the exchange of the rest. Now let us consider the effect of $\{\alpha_{k1}..\alpha_{kp}\}\leftrightarrow\{\alpha_{l1}..\alpha_{lp}\}$ on C. After the exchange, C_q can either change sign or stay invariant depending on whether an odd or even number (out of q) antisymmetric indices are contained in the specified triplets. In other words upon $\{\alpha_{k1}..\alpha_{kp}\}\leftrightarrow\{\alpha_{l1}..\alpha_{lp}\}$ we have

$$C\to\sum_{q=3,\text{odd}}^{3p}\eta_qC_q,\tag{A12}$$

where $\eta_q=\pm1$. Since Eq.(A8) is not satisfied, all η_q must not simultaneously be -1. Now consider a new C

$$C'\equiv\frac{1}{2}\Big[C-\sum_{q=3,\text{odd}}^{3p}\eta_qC_q\Big].\tag{A13}$$

It is obvious that upon $\{\alpha_{k1}..\alpha_{kp}\}\leftrightarrow\{\alpha_{l1}..\alpha_{lp}\}$ $C'\to-C'$. Moreover by construction C' only contains those C_q whose $\eta_q=-1$. Now use C' as the starting C and repeat the above operation until we reach a final C' for which Eq.(A8) holds for all triplet exchanges and for all (ij). Since at each stage of obtaining C' certain C_q are projected out, there must be missing components in the final C. However we have already proven that any C that satisfy Eq.(A8) for all (ij) pair must lead to the solution $\chi\sim\Psi_m$. However, Ψ_m contains all components for all pair (ij). Consequently we have reached a contradiction. Therefore it must be possible to make Eq.(A8) hold true for all pairs (ij) for any ground state solution satisfying Eq.(A3).

The proof can be generalized to any m by assigning m groups of indices. Thus, we have proven that Ψ_m is the unique zero-energy state of Eq.(37).

Appendix B: Summary of SU(3) algebra and representation theory

1. Algebra

The SU(3) group is the one with which people are very familiar besides SU(2). This note will not be a thorough review of Lie algebra but focuses on what we shall need in the paper. The generators of SU(3)

$$\begin{aligned}&T_1=\frac{1}{2}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\ T_2=\frac{1}{2}\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix},\\&T_3=\frac{1}{2}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\ T_4=\frac{1}{2}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\\&T_5=\frac{1}{2}\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix},\ T_6=\frac{1}{2}\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},\\&T_7=\frac{1}{2}\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix},\ T_8=\frac{1}{\sqrt{12}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}\end{aligned}$$

satisfy

$$Tr(T_a T_b) = \frac{1}{2}\delta_{ab} \tag{B1}$$

in the standard convention. The Cartan subalgebra contains T_3 and T_8. Denote them by H_1 and H_2 respectively. The simply roots of SU(3) can be obtained as

$$\alpha^1 = (\frac{1}{2}, \frac{\sqrt{3}}{2}),\ \alpha^2 = (\frac{1}{2}, -\frac{\sqrt{3}}{2}) \tag{B2}$$

All positive roots of SU(3) are given by α^1, α^2, and $\alpha^1 + \alpha^2$. The generators correspond to the positive roots are given by

$$\begin{aligned} E_{\alpha^1} &= \frac{1}{\sqrt{2}}(T_4 + iT_5), \\ E_{\alpha^2} &= \frac{1}{\sqrt{2}}(T_6 - iT_7), \\ E_{\alpha^1+\alpha^2} &= \frac{1}{\sqrt{2}}(T_1 + iT_2) \end{aligned} \tag{B3}$$

These generators are the raising operators in SU(3). Their lowing operators are the Hermitian conjugates of themselves, *i.e.*

$$\begin{aligned} E_{-\alpha^1} &= \frac{1}{\sqrt{2}}(T_4 - iT_5), \\ E_{-\alpha^2} &= \frac{1}{\sqrt{2}}(T_6 + iT_7), \\ E_{-\alpha^1-\alpha^2} &= \frac{1}{\sqrt{2}}(T_1 - iT_2) \end{aligned} \tag{B4}$$

The algebra is given by

$$[E_{\alpha^1}, E_{-\alpha^1}] = E_3^1 = \alpha^1 \cdot H \tag{B5}$$

$$[E_{\alpha^2}, E_{-\alpha^2}] = E_3^2 = \alpha^2 \cdot H \tag{B6}$$

$$[E_{\alpha^1}, E_{\alpha^2}] = \frac{1}{\sqrt{2}} E_{\alpha^1+\alpha^2} \tag{B7}$$

$$[E_{-\alpha^1}, E_{\alpha^1+\alpha^2}] = \frac{1}{\sqrt{2}} E_{\alpha^2} \tag{B8}$$

$$[E_{-\alpha^2}, E_{\alpha^1+\alpha^2}] = -\frac{1}{\sqrt{2}} E_{\alpha^2} \tag{B9}$$

The fundamental weight is defined by

$$\frac{2\alpha^i \cdot \mu^j}{|\alpha^i|^2} = \delta_{ij} \tag{B10}$$

In SU(3), μ^i's are given by

$$\mu^1 = (\frac{1}{2}, \frac{\sqrt{3}}{6}),\ \mu^2 = (\frac{1}{2}, -\frac{\sqrt{3}}{6}) \tag{B11}$$

The representation whose highest weight is the fundamental weight is called the fundamental representation. Since the rank of SU(N) group is $N-1$. The number of simply root and that of the fundamental weight are also $N-1$. The highest weight μ in *any* SU(3) representation is given by $\mu = p\mu^1 + q\mu^2$. p and q are called Dynkin coefficients, which are unique for every representation. Therefore, SU(3) representations are denoted by (p, q). The total number of the Casimir operators of SU(N) is also equal to its rank. We shall pay our attention to the quadratic Casimir operator only. It is defined by

$$C = \sum_a T_a T_a \tag{B12}$$

We can compute it to be

$$C = H_1^2 + H_2^2 + \sum_{all\ positive\ roots} E_\alpha E_{-\alpha} \tag{B13}$$

Using Eq.(B5)(B6)(B7)(B8)(B9) and the highest weight method, we compute the quadratic Casimir for SU(3) (p, q) representation:

$$C[p, q] = \frac{1}{3}(p^2 + pq + q^2 + 3p + 3q) \tag{B14}$$

The dimension of (p, q) representation is given by

$$D[p, q] = \frac{(p+1)(q+1)(p+q+2)}{2} \tag{B15}$$

In the context, we introduce the SU(3) algebra in favor of particle physics. Namely, the SU(3) algebra is given in the Gell-Mann notation. There is another basis which is also very interesting and useful in certain problems[14].

If we define

$$(A_k^i)_{\mu\nu} = \delta_{i\nu}\delta_{k\mu} - \frac{1}{3}\delta_{ik}\delta_{\mu\nu} \tag{B16}$$

where $i, k, \mu, \nu = 1, 2, 3$, having the following properties:

$$A_k^i = (A_i^k)^\dagger \tag{B17}$$

$$A_1^1 + A_2^2 + A_3^3 = 0 \tag{B18}$$

in which we know that there are only 8 independent generators. It can be checked that they satisfy the following commutation relations:

$$[A_k^i, A_l^j] = \delta_l^i A_k^j - \delta_k^j A_l^i \tag{B19}$$

These 8 independent generators form the SU(3) algebra. The Cartan subalgebra is given by

$$\begin{aligned} h_1 &= \frac{1}{2}(A_1^1 - A_2^2) = T_3 \\ h_2 &= \frac{1}{2}(A_2^2 - A_3^3) = -\frac{1}{2}T_3 + \frac{\sqrt{3}}{2}T_8 \end{aligned} \tag{B20}$$

This notation is so-called Okubo's notation. The relation between Okubo notation and the Gell-Mann notation is given as the following. Denote

$$\begin{aligned} &T_1 = I_1,\ T_2 = I_2,\ T_3 = I_3 \\ &T_4 = K_1,\ T_5 = K_2 \\ &T_6 = L_1,\ T_7 = L_2 \\ &T_8 = M \end{aligned} \tag{B21}$$

and

$$I_\pm = I_1 \pm iI_2$$
$$K_\pm = K_1 \pm iK_2$$
$$L_\pm = L_1 \pm iL_2, \tag{B22}$$

and the A^i_k can be written as

$$A^1_1 = I_3 + \frac{1}{3}\sqrt{3}M,\ A^2_1 = I_+,\ A^1_2 = I_-,$$
$$A^2_2 = -I_3 + \frac{1}{3}\sqrt{3}M,\ A^3_1 = K_+,\ A^1_3 = K_-,$$
$$A^3_3 = -\frac{2}{3}\sqrt{3}M,\ A^3_2 = L_+,\ A^2_3 = L_- \tag{B23}$$

On the other hand, it is possible to obtain the transformation between I, K, and L: Define

$$P_i = e^{i\pi I_2}$$
$$P_k = e^{i\pi K_2}$$
$$P_l = e^{i\pi L_2}, \tag{B24}$$

Then

$$P_i^{-1} I_\pm P_i = -I_\mp,\ P_i^{-1} K_\pm P_i = L_\pm,$$
$$P_k^{-1} I_\pm P_k = L_\mp,\ P_k^{-1} K_\pm P_k = -K_\mp,$$
$$P_l^{-1} I_\pm P_l = K_\pm, P^{-1} K_\pm P_l = -I_\pm,$$
$$P^{-1} L_\pm P_i = -K_\pm,\ P_k^{-1} L_\pm P_k = -I_\mp,$$
$$P_l^{-1} L_\pm P_l = -L_\mp \tag{B25}$$

2. Representation in the X-L$_3$ basis

The X operator satisfies the following equation

$$X|(p,q)jLL_z\rangle = 2p + q - 6j|(p,q)jLL_z\rangle \tag{B26}$$

j ranges from $0, \frac{1}{2}$,..to $\frac{p+q}{2}$. $j = 0$ is the highest X state. j is a quantum number on the X-axis. However, due to degeneracy, for a certain j, it could be many L. The value of L is given by $|j - \frac{q}{2}|...j + \frac{q}{2}$. Using this basis the SU(3) Clebsch-Gordan coefficients (isofactor) are given by

$$\langle (p_1,0)M,(p_2,0)N||(p,q)jL\rangle$$
$$= (-1)^q \sqrt{\frac{(p+1)(q+1)!(p+q+L-M-N+1)!}{(p_1-q)!(p_2-q)!}}$$
$$\times \sqrt{\frac{(p+q-L-M-N)!(2M+2N-q+1)!}{(p_1-2M)!(p_2-2N)!}}$$
$$\times \frac{\sqrt{(2M+1)!(2N+1)!}}{(p+q+1)!}$$
$$\times \sum_{I'_1+I'_2=\frac{q}{2}} (-1)^{2I'_2} \frac{(p_1-2I'_1)!(p_2-2I'_2)!}{(2I'_1)!(2I'_2)!(2M-2I'_1)!(2N-2I'_2)!}$$
$$\times \begin{Bmatrix} I'_1 & I'_2 & \frac{q}{2} \\ M-I'_1 & N-I'_2 & j \\ M & N & L \end{Bmatrix} \tag{B27}$$

where $j = M + N - \frac{q}{2}$ and we use the Wigner 9-j symbol. If $L = M + N$, namely $L = j + \frac{q}{2}$ case, the Wigner 9-j symbol has a simpler form:

$$\begin{Bmatrix} I'_1 & I'_2 & \frac{q}{2} \\ M-I'_1 & N-I'_2 & j \\ M & N & M+N \end{Bmatrix}$$
$$= \frac{1}{\sqrt{(q+1)(2M+1)(2N+1)(2j+1)}} \tag{B28}$$

The CG coefficient becomes

$$\langle (p_1,0)M,(p_2,0)N||(p,q)jL\rangle$$
$$= (-1)^q \sqrt{\frac{(p+1)q!(p+q-2M-2N)!}{(p_1-q)!(p_2-q)!}}$$
$$\times \sqrt{\frac{(2M)!(2N)!(2M+2N-q)!}{(p+q+1)!(p_1-2M)!(p_2-2N)!}}$$
$$\times \sum_{I'_1+I'_2=\frac{q}{2}} (-1)^{2I'_2} \frac{(p_1-2I'_1)!(p_2-2I'_2)!}{(2I'_1)!(2I'_2)!(2M-2I'_1)!(2N-2I'_2)!} \tag{B29}$$

This result can be compared with the CG coefficient to the highest weight:

$$\langle (p_1,0)M,(p_2)N||(p,q)0,\frac{q}{2}\rangle$$
$$= (-1)^{q+2M} \frac{(p+1)!q!}{(p+q+1)!(p_1-q)!(p_2-q)!}$$
$$\times \frac{(p_1-2M)!(p_2-2N)!}{(2M)!(2N)!} \delta_{M+N,\frac{q}{2}} \tag{B30}$$

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